Graphs and Combinatorics 2, 95–100 (1986)

Graphs and Combinatorics

Decomposition of the Complete r-Graph into Complete r-Partite r-Graphs*

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Abstract. For $n \geq r \geq 1$, let $f_r(n)$ denote the minimum number q, such that it is possible to partition all edges of the complete r-graph on n vertices into q complete r-partite r-graphs. Graham and Pollak showed that $f_2(n) = n - 1$. Here we observe that $f_3(n) = n - 2$ and show that for every fixed $r \geq 2$, there are positive constants $c_1(r)$ and $c_2(r)$ such that $c_1(r) \leq f_r(n) \cdot n^{-[r/2]} \leq c_2(r)$ for all $n \geq r$. This solves a problem of Aharoni and Linial. The proof uses some simple ideas of linear algebra.

1. Introduction

For $n \geq r \geq 1$, let $f_r(n)$ denote the minimum number q, such that it is possible to partition all edges of the complete r-uniform hypergraph on n vertices into q pairwise edge-disjoint complete r-partite r-uniform hypergraphs.

Obviously, $f_1(n) = 1$. Graham and Pollak ([3, 4], see also [2, 5]) proved that $f_2(n) = n - 1$ for all $n \geq 2$. Simple proofs for this result were found by Tverberg [7] and Peck [6].

Aharoni and Linial [1] raised the natural problem of determining or estimating $f_r(n)$ for $r > 2$. In particular they asked if $f_r(n)$ is a nonlinear function of n, for some fixed $r > 2$.

In this note we answer this question in the affirmative by proving the following theorem, that determines the asymptotic behavior of $f_r(n)$ for every fixed r as n tends to infinity.

Theorem 1.1. *For every fixed $r \geq 1$, there are two positive constants $c_1 = c_1(r)$ and $c_2 = c_2(r)$ such that*

$$c_1 \cdot n^{[r/2]} \leq f_r(n) \leq c_2 \cdot n^{[r/2]}$$

for all $n \geq r$.

The lower bound is proved using some simple ideas of linear algebra. The method is similar to the one used by Tverberg [7] and by Graham and Pollak [3, 4], for determining $f_2(n)$. The upper bound is established by a recursive construction.

* Research supported in part by Air Force Contract OSR 82-0326 and by Allon Fellowship.

It is worth noting that our construction supplies the exact value of $f_3(n) = n - 2$ for all $n \geq 3$.

2. The Lower Bound

We start with the following easy observation.

Lemma 2.1. *For every $n \geq r \geq 2$*

$$f_r(n) \geq f_{r-1}(n-1).$$

Proof. Suppose all edges of the complete r-uniform hypergraph on a set $N = \{1, 2, \ldots, n\}$ of n vertices are partitioned into $q = f_r(n)$ r-partite r-graphs ($= r$ uniform hypergraphs) $H^1, H^2, \ldots, H^q$. Let E_i denote the set of edges of H^i and put $\bar{E}_i = \{e - \{n\}: e \in E_i, n \in e\}$. Clearly each nonempty $\bar{E}_i$ is the set of edges of a complete $(r-1)$-partite $(r-1)$-graph. Moreover, the set of all nonempty $\bar{E}_i$'s forms a decomposition of all edges of the complete $r-1$-uniform hypergraph on the $n-1$ vertices $N - \{n\}$. Hence $f_{r-1}(n-1) \leq q = f_r(n)$, as needed. □

In view of Lemma 2.1, the lower bound in Theorem 1.1 for odd values of r follows from the lower bound for even values of r, which we prove next.

Lemma 2.2. *For all $n \geq 2k \geq 2$*

$$f_{2k}(n) \geq 2 \cdot \frac{\binom{n}{k} - \binom{n}{k-1} - \binom{n}{k-3} - \cdots - \binom{n}{k+1-2\cdot\lceil k/2 \rceil}}{\binom{2k}{k}}.$$

Proof. Let $\underline{K} = \{K \subset N: |K| = k\}$ be the set of all $\binom{n}{k}$ k-subsets of $N = \{1, 2, \ldots, n\}$ and associate each $K \in \underline{K}$ with a variable x_K. Let H be a complete $2k$-partite $2k$-graph, whose (pairwise disjoint) vertex classes $V_1, V_2, \ldots, V_{2k}$ are subsets of N. By definition, the edges of H are all $2k$-subsets $A \subset N$, such that $|A \cap V_i| = 1$ for $1 \leq i \leq 2k$. We define, for each such H, a quadratic form $Q(H)$ in the variables $\{x_K: K \in \underline{K}\}$ as follows.

$Q(H) = \sum \{L_A(H) \cdot L_B(H): A,\ B \subset \{1, 2, \ldots, 2k\},\ |A| = |B| = k,\ A \cap B = \varnothing,\ 1 \in A\}$, where, for $C \subset \{1, 2, \ldots, 2k\}, |C| = k$,

$$L_C(H) = \sum \{x_K: K \in \underline{K}, |K \cap V_c| = 1 \text{ for all } c \in C\}.$$

Thus, $Q(H)$ is a sum of $\frac{1}{2}\binom{2k}{k}$ products of the form $L_A(H) \cdot L_B(H)$, in which each factor is a linear combination of the x_K's.

Put $q = f_{2k}(n)$, and suppose the edges of the complete r-graph on N are partitioned into q r-partite r-graphs $H^1, H^2, \ldots, H^q$. One can easily check that

$$\sum_{i=1}^{q} Q(H^i) = \sum \{x_K \cdot x_L: K, L \in \underline{K}, K \cap L = \varnothing\}. \tag{2.1}$$

Indeed, if $K, L \in \underline{K}$ and $K \cap L = \varnothing$ then the product $x_K \cdot x_L$ appears only in $Q(H^i)$, where H^i is the unique H^j containing $K \cup L$ as an edge, and if $K \cap L \neq \varnothing$, then $x_k \cdot x_L$ appears in no $Q(H^i)$.

We next claim that

$$\begin{cases} \sum\{x_K \cdot x_L \colon K, L \in \underline{K}, K \cap L = \varnothing\} \\ = \dfrac{1}{2} \sum_{i=0}^{k} (-1)^i \sum_{A \subset N, |A| = i} \left(\sum_{K \in \underline{K}, A \subset K} x_K \right)^2. \end{cases} \tag{2.2}$$

Indeed, if $K, L \in \underline{K}$ and $|K \cap L| = j$, $(0 \leq j < k)$, then the coefficient of $x_K \cdot x_L$ in the right hand side of (2.2) is $\sum_{i=0}^{j} (-1)^i \binom{j}{i}$, which is 1 if $j = 0$ and 0 if $j > 0$. For $K = L$, the coefficients of x_K^2 in the right hand side of (2.2) is $\frac{1}{2} \sum_{i=0}^{k} (-1)^i \binom{k}{i} = 0$. Thus (2.2) holds.

Substituting (2.2) and the definition of the $Q(H^i)$'s into (2.1) we conclude that

$$\begin{cases} \sum_{i=1}^{q} \sum \{L_A(H^i) \cdot L_B(H^i) \colon A, B \subset \{1, \ldots, 2k\}, |A| = |B| = k, A \cap B = \varnothing, 1 \in A\} \\ \qquad = \dfrac{1}{2} \sum_{i=0}^{k} (-1)^i \sum_{A \subset N, |A| = i} \left(\sum_{K \in \underline{K}, A \subset K} x_K \right)^2 \end{cases} \tag{2.3}$$

Let V be the linear subspace of the real $\binom{n}{k}$-dimensional space of the x_K's determined by the following set of

$$\frac{1}{2}\binom{2k}{k} \cdot q + \binom{n}{k-1} + \binom{n}{k-3} + \cdots + \binom{n}{k+1-2\lceil k/2 \rceil}$$

linear equations.

$$\begin{cases} L_A(H^i) = 0 \text{ for all } 1 \leq i \leq q \text{ and } A \subset \{1, 2, \ldots, 2k\}, |A| = k, 1 \in A. \\ \sum_{K \in \underline{K}, A \subset K} x_K = 0 \text{ for all } A \subset N, |A| \in \{k-1, k-3, \ldots, k+1-2\lceil k/2 \rceil\} \end{cases} \tag{2.4}$$

We claim that V is the zero subspace. Indeed, suppose $\{\bar{x}_k \colon K \in \underline{K}\} \in V$. Then $\bar{x}_K$ satisfies (2.4), and in view of (2.3) we conclude that

$$0 = \frac{1}{2} \cdot (-1)^k \left\{ \sum_{K \in \underline{K}} \bar{x}_K^2 + \sum_{A \subset N, |A| = k-2} \left(\sum_{K \in \underline{K}, A \subset K} \bar{x}_K \right)^2 + \cdots \right\},$$

and hence $\bar{x}_K = 0$ for all $K \in \bar{K}$.

Therefore, the number of linear equations in the system (2.4) is at least $\binom{n}{k}$ and the assertion of Lemma 2.2 follows. □

Combining Lemmas 2.1 and 2.2 we obtain

Corollary 2.3. *For every fixed* $r \geq 1$, $f_r(n) \geq c_r \cdot n^{[r/2]} \cdot (1 + o(1))$ as $n \to \infty$, *where*

$$c_r = \frac{2[r/2]!}{(2[r/2])!}.$$

Remarks.
1) Lemma 2.2 with $k = 1$ reduces to Graham-Pollak's result; $f_2(n) \geq n - 1$, (which is, of course, sharp).
2) Lemma 2.1 with $r = 3$ asserts $f_3(n) \geq f_2(n-1) = n - 2$. As shown in the next section this result is also sharp.
3) A trivial lower bound for $f_r(n)$ is $f_r(n) \geq \binom{n}{r} \Big/ \left(\frac{n}{r}\right)^r$, since the number of edges of any complete r-partite r-graph on n vertices is not greater than $(n/r)^r$. This trivial bound is much weaker than the one proved above for all $r = o(n)$, but is better for, e.g., $r = [n/2]$.

3. The Upper Bound

In this section we prove the upper bound for $f_r(n)$ given in Theorem 1.1, using some simple recursive constructions. We first determine $f_3(n)$ for all $n \geq 3$.

Lemma 3.1. *For all* $n \geq 3$

$$f_3(n) = n - 2.$$

Proof. By Lemma 2.1 and Graham-Pollak's result

$$f_3(n) \geq f_2(n-1) = n - 2.$$

We prove that $f_3(n) \leq n - 2$ by induction on n. For $n = 2, 3$ the result is trivial. Assuming the result for all n', $n' < n$, we prove it for n, $(n > 3)$. Put $N = \{1, 2, \ldots, n\}$ and $N_i = \{2i-1, 2i\}$ for $1 \leq i \leq [n/2]$. For odd n define also $N_{\lceil n/2 \rceil} = \{n\}$. We claim that

$$f_3(n) \leq [n/2] + f_3(\lceil n/2 \rceil). \tag{3.1}$$

Indeed, put $q = f_3(\lceil n/2 \rceil)$ and let $H^1, \ldots, H^q$ be a decomposition of the complete 3-graph on $\lceil n/2 \rceil$ vertices $\{1, 2, \ldots, \lceil n/2 \rceil\}$ into q complete 3-partite 3-graphs. For $1 \leq i \leq q$, let V_1^i, V_2^i and V_3^i denote the vertex-classes of H^i. Let $\bar{H}^i$ be the 3-partite 3-graph whose vertex classes are $\cup\{N_j : j \in V_1^i\}$, $\cup\{N_j : j \in V_2^i\}$ and $\cup\{N_j : j \in V_3^i\}$. For $1 \leq j \leq [n/2]$, let $\bar{H}^{q+j}$ be the 3-partite 3-graph whose vertex classes are $\{2i-1\}$, $\{2i\}$ and $N - \{2i-1, 2i\}$. One can easily check that the hypergraphs $\{\bar{H}^i\}_{i=1}^{q+[n/2]}$ form a dcomposition of all edges of the compolete 3-graph on N into 3-partite 3-graphs. This establishes (3.1). Hence, by the induction hypothesis,

$$f_3(n) \leq [n/2] + f_3(\lceil n/2 \rceil) \leq [n/2] + \lceil n/2 \rceil - 2 = n - 2. \qquad \square$$

Let N_1 and N_2 be two disjoint sets of vertices, and let H_i be an r_i-graph on N_i, $(1 \leq i \leq 2)$. We denote by $H_1 + H_2$ the $(r_1 + r_2)$-graph on $N_1 \cup N_2$ whose edges are all edges $e_1 \cup e_2$, where e_i is an edge of H_i $(i = 1, 2)$. One can easily check that if H_i is a complete r_i-partite r_i-graph then $H_1 + H_2$ is a complete $(r_1 + r_2)$-partite $(r_1 + r_2)$-graph. For notational convenience let us agree that $f_0(n) = 1$ for all n.

Lemma 3.2. *Suppose* $n \geq r \geq 4$, *then*

$$f_r(n) \leq \sum_{i=0}^{r} f_i([n/2]) \cdot f_{r-i}(\lceil n/2 \rceil).$$

Proof. Put $N = \{1, 2, \ldots, n\}$, $N_1 = \{1, 2, \ldots, [n/2]\}$, $N_2 = \{[n/2] + 1, \ldots, n\}$. For $0 \le i \le r$ let $\underline{H}^i$ be a family of $f_i([n/2])$ complete i-partite i-graphs that decompose the complete i-graph on N_1. ($\underline{H}^0$ consists of one graph whose only edge is the empty edge.) Similarly, let $\underline{G}^j$ be a family of $f_j(\lceil n/2\rceil)$ complete j-partite j-graphs that decompose the complete j-graph on N_2 $(0 \le j \le r)$. Define a family $\underline{F}$ of $\sum_{i=0}^{r} f_i([n/2]) \cdot f_{r-i}(\lceil n/2\rceil)$ complete r-partite r-graphs on N by

$$\underline{F} = \bigcup_{i=0}^{r} \{H^i + G^{r-i} \colon H^i \in \underline{H}^i, G^{r-i} \in \underline{G}^{r-i}\}.$$

One can easily check that the members of $\underline{F}$ form a decomposition of the complete r-graph on N. This completes the proof. $\square$

We can now prove the upper bound for $f_r(n)$ given in Theorem 1.1 by double induction on r and n. Since $f_r(n)$ is a monotone increasing function of n, it is enough to prove it when n is a power of 2, which we assume, for convenience. By Lemma 3.1 (and trivial constructions for $r \le 2$) $f_r(n) \le c_r \cdot n^{[r/2]}$ for $r = 0, 1, 2, 3$ and every n, where $c_0 = c_1 = c_2 = c_3 = 1$. Clearly, if $n < r$ then $f_r(n) \le c_r \cdot n^{[r/2]}$ for every positive c_r. Assuming that

$$f_{r'}(n') \le c_{r'} \cdot n'^{[r'/2]} \tag{3.2}$$

for all $r' < r$ and all $n' = 2^j$, and for $r' = r$ and $2^j = n' < n = 2^i$, we prove that if c_r is properly chosen then (3.2) holds also for (r, n). Indeed, by Lemma 3.2 and the induction hypothesis

$$\begin{aligned} f_r(n) &\le \sum_{i=0}^{r} f_i(n/2) f_{r-i}(n/2) \le \sum_{i=0}^{r} c_i \cdot c_{r-i} (n/2)^{[i/2]+[(r-i)/2]} \\ &\le \sum_{i=0}^{r} c_i \cdot c_{r-i} \cdot (n/2)^{[r/2]} = \frac{1}{2^{[r/2]}} \left(2c_r + \sum_{i=1}^{r-1} c_i \cdot c_{r-i} \right) n^{[r/2]}. \end{aligned}$$

Hence, if we define the c_r's by

$$\begin{aligned} &c_0 = c_1 = c_2 = c_3 = 1 \\ &\text{and } c_r \cdot (2^{[r/2]} - 2) = \sum_{i=1}^{r-1} c_i \cdot c_{r-i} \text{ for } r \ge 4, \end{aligned} \tag{3.3}$$

then $f_r(n) \le c_r \cdot n^{[r/2]}$ for every r and every $n = 2^i$. This implies the validity of the upper bound for $f_r(n)$ given in Theorem 1.1.

Remarks.

1) One can easily check that the constants $\{c_r\}_{r=0}^{\infty}$ defined by (3.3) satisfy

$$c_r \le \frac{8^{r-1}}{\lceil r/2\rceil !}.$$

By a somewhat more careful analysis we can show that the construction described above implies that for every fixed $k \ge 1$ $f_{2k}(n) \le \frac{1}{k!} \cdot n^k(1 + o(1))$, as $n \to \infty$. This should be compared to the lower bound

$$f_{2k}(n) \geq \frac{2k!}{(2k)!} \cdot n^k(1 + o(1))$$

given in Corollary 2.3.

2) It would be interesting to determine $f_r(n)$ precisely for $r > 3$, or to improve our estimates. In particular, Lemma 2.2 and Lemma 3.2 for $r = 4$ imply that

$$\frac{1}{6}(n^2 - 3n) \leq f_4(n) \leq \frac{1}{2}(n^2 - 5n + 6)$$

for all n. It would be interesting to decide which of these two bounds is closer to the truth.

Acknowledgement. I would like to thank P. Frankl for fruitful discussions.

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Received: November 15, 1985
Revised: December 17, 1985